Int. J. Solids Structures, 1966, Vol 2, pp. 351 to 360. Pergamon Press Ltd. Printed in Great Britain

THICKNESS-TWIST AND FACE-SHEAR VIBRATIONS OF A CONTOURED CRYSTAL PLATE

JEFFREY L. BLEUSTEIN
Bell Telephone Laboratories, Inc., Whippany, New Jersey

Abstract—Solutions are obtained for the thickness-twist and face-shear vibrations in a crystal plate with a linearly varying thickness. The results are compared to those obtained for a plate of constant thickness. It is found that the frequency of the lowest thickness-twist mode of a linearly tapered plate is only slightly higher than the frequency of the lowest thickness-twist mode of a uniform plate of the same maximum thickness. It is also found that the frequencies of the face-shear modes in a tapered plate differ very little from those in a uniform plate.

INTRODUCTION AND SUMMARY

INTEREST in the analysis of the vibrations of contoured plates (plates of varying thickness) stems from at least two sources. In the first place a study of these vibrations might point the way to the design of new transducer devices based on special features of the frequency spectrum. Secondly, the results should be useful for determining the effects of small manufacturing errors on the frequency spectrum of a plate of nominally constant thickness.

Previous studies [1], [2], of the vibrations of contoured crystal plates have dealt with the coupled thickness-shear and flexural vibrations in which there is a nonzero component of displacement in the direction along which the thickness varies. In the present paper, an investigation is made of modes of motion of a contoured plate in which the displacements are all perpendicular to the direction along which the thickness varies. The particular modes under consideration are the thickness-twist and the face-shear modes, and as Mindlin and Gazis have pointed out [3], [4], they are modes of technological interest since they can be strongly excited piezoelectrically in a quartz plate.

Thickness-twist vibrations are contained in the equations of the approximate theory which is the starting point for the analysis in [1] and [2]. However a more general approximate theory has been developed [5] which takes into account the coupling of flexural, extensional and face-shear deformations with each other and with the lowest thickness orders of thickness-shear and thickness-twist deformations. In the present paper, the equations and the notation of the more general theory are employed.

The equations are specialized to the case of an infinite strip which has a linear taper in the direction perpendicular to its infinite dimension. In the sequel this strip will be referred to simply as a plate. It is found that the frequency of the lowest thickness-twist mode of a linearly tapered plate is only slightly higher than the frequency of the lowest thickness-twist mode of a uniform plate whose thickness is equal to the maximum thickness of the tapered plate. In addition, the frequencies of the thickness-twist overtones are spread out more in a tapered plate than they are in a uniform plate. These results can be associated with the fact that the fundamental thickness-twist motion is localized at the thick edge of the plate whereas, for the overtones, a greater portion of the plate is involved in the motion. It is also found that the frequencies of the face-shear modes in a tapered plate differ very little from those in a uniform plate. This result is consistent with the solution for a uniform plate where it is found that the frequencies of the face-shear modes are independent of the thickness of the plate.

GOVERNING EQUATIONS

The stress–strain relation of a rotated-Y-cut quartz plate, referred to a rectangular cartesian coordinate system, x_1, x_2, x_3, with x_1 a diagonal axis and $x_2 = 0$ the middle plane of the plate, exhibits monoclinic symmetry [6]. An abbreviated notation is employed whereby a pair of indices which range over the integers 1, 2, 3 is replaced by a single index ranging over the integers 1, 2, 3, 4, 5, 6 according to the scheme in Table 1:

TABLE 1. ABBREVIATED NOTATION

Replace ij =	11	22	33	23 or 32	31 or 13	12 or 21
by p =	1	2	3	4	5	6

With this notation the governing equations [5] for the free vibrations of a plate of thickness $2h$ are:

Stress equations of motion

$$\begin{aligned}
\tau^{(0)}_{1,1}+\tau^{(0)}_{5,3} &= 2h\rho\ddot{u}^{(0)}_1,\\
\tau^{(0)}_{6,1}+\tau^{(0)}_{4,3} &= 2h\rho\ddot{u}^{(0)}_2,\\
\tau^{(0)}_{5,1}+\tau^{(0)}_{3,3} &= 2h\rho\ddot{u}^{(0)}_3,\\
\tau^{(1)}_{1,1}+\tau^{(1)}_{5,3}-\tau^{(0)}_6 &= \tfrac{2}{3}\rho h^3\ddot{u}^{(1)}_1,\\
\tau^{(1)}_{5,1}+\tau^{(1)}_{3,3}-\tau^{(0)}_4 &= \tfrac{2}{3}\rho h^3\ddot{u}^{(1)}_3,
\end{aligned} \tag{1}$$

where, $a\,(a = 1, 3)$ denotes the partial derivative, $\partial/\partial x_a$, and the dot denotes partial differentiation with respect to time;

Constitutive equations

$$\begin{aligned}
\tau^{(0)}_p &= 2h\bar{c}^*_{pq}s^{(0)}_q \qquad p, q = 1, 2, \ldots, 6,\\
\tau^{(1)}_a &= \tfrac{2}{3}h^3\gamma_{ab}s^{(1)}_b \qquad a, b = 1, 3, 5,
\end{aligned} \tag{2}$$

where

$$\bar{c}^*_{pq} = k^{\mu}_{(p)}k^{\nu}_{(q)}\bar{c}_{pq}, \qquad \text{no sum on } p \text{ or } q,$$

$$\bar{c}_{pq} = c_{pq}-c_{p2}c_{2q}/c_{22},$$

$$c_{15} = c_{25} = c_{35} = c_{45} = c_{16} = c_{26} = c_{36} = c_{46} = 0,$$

$$\mu = \cos^2(p\pi/2), \qquad \nu = \cos^2(q\pi/2),$$

$$k^2_4 = \pi^2\{c_{22}+c_{44}-[(c_{22}-c_{44})^2+4c^2_{24}]^{\frac{1}{2}}\}/24\bar{c}_{44},$$

$$k_6^2 = \pi^2/12,$$

$$\gamma_{ab} = \bar{c}_{ab} - \bar{c}_{4a}\bar{c}_{b4}/c_{44} \qquad a, b = 1, 3,$$

$$\gamma_{55} = c_{55} - c_{56}^2/c_{66};$$

Strain–displacement relations

$$\begin{aligned} s_1^{(0)} &= u_{1,1}^{(0)}, & s_1^{(1)} &= u_{1,1}^{(1)}, \\ s_3^{(0)} &= u_{3,3}^{(0)}, & s_3^{(1)} &= u_{3,3}^{(1)}, \\ s_4^{(0)} &= u_{2,3}^{(0)} + u_3^{(1)}, & & \\ s_5^{(0)} &= u_{3,1}^{(0)} + u_{1,3}^{(0)}, & s_5^{(1)} &= u_{1,3}^{(1)} + u_{3,1}^{(1)}; \\ s_6^{(0)} &= u_{2,1}^{(0)} + u_1^{(1)}, & & \end{aligned} \tag{3}$$

Boundary conditions

For the free edges of a rectangular plate of length $2l$ and width $2w$,

$$\begin{aligned} \tau_1^{(0)} = \tau_5^{(0)} = \tau_6^{(0)} = \tau_1^{(1)} = \tau_5^{(1)} = 0 \text{ on } x_1 = \pm w, \\ \tau_3^{(0)} = \tau_5^{(0)} = \tau_4^{(0)} = \tau_3^{(1)} = \tau_5^{(1)} = 0 \text{ on } x_3 = \pm l. \end{aligned} \tag{4}$$

MOTION INDEPENDENT OF x_1

We consider the case in which $h = h(x_3)$ and in which

$$\begin{aligned} u_1^{(0)} &= \bar{u}(x_3)e^{i\omega t}, & u_2^{(0)} &= u_3^{(0)} = 0, \\ u_1^{(1)} &= \psi(x_3)e^{i\omega t}, & u_3^{(1)} &= 0. \end{aligned} \tag{5}$$

The nonzero components of displacement are the face-shear, $u_1^{(0)}$, and the thickness-twist, $u_1^{(1)}$. These displacements are illustrated in Fig. 1. The corresponding nonzero strain components are

$$s_5^{(0)} = \bar{u}'e^{i\omega t}, \qquad s_6^{(0)} = \psi e^{i\omega t}, \qquad s_5^{(1)} = \psi' e^{i\omega t} \tag{6}$$

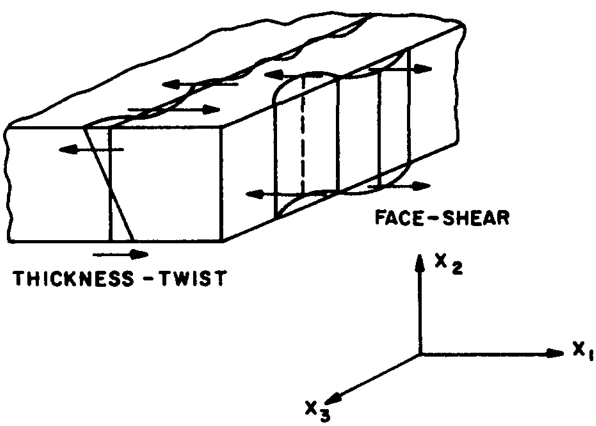

FIG. 1. Thickness-twist and face-shear displacements.

where $' \equiv d/dx_3$. The associated nonzero stress-resultants are

$$\tau_5^{(0)} = 2h(c_{55}\bar{u}' + k_6 c_{56}\psi)e^{i\omega t},$$
$$\tau_6^{(0)} = 2hk_6(c_{56}\bar{u}' + k_6 c_{66}\psi)e^{i\omega t}, \tag{7}$$
$$\tau_5^{(1)} = \tfrac{2}{3}h^3\gamma_{55}\psi' e^{i\omega t}.$$

For the AC-cut of quartz the elastic coefficient c_{56} is zero, and equations (7) are considerably simplified. For the AT-cut of quartz c_{56} is small relative to c_{55} and c_{66}, and as a first approximation we can take it identically equal to zero. It is evident from the work in [3] that this assumption should not prohibit the matching of theoretical with experimental results. With this assumption, inserting (7) into (1) we obtain

$$\bar{u}'' + \frac{h'}{h}\bar{u}' + \bar{\beta}^2\bar{u} = 0, \tag{8a}$$

$$\psi'' + \frac{3h'}{h}\psi' + \left(\beta^2 - \frac{\gamma^2}{h^2}\right)\psi = 0, \tag{8b}$$

where

$$\beta^2 \equiv \rho\omega^2/c_{55}, \qquad \bar{\beta}^2 \equiv \rho\bar{\omega}^2/c_{55}, \qquad \gamma^2 \equiv 3k_6^2 c_{66}/c_{55}.$$

We note that with the assumption $c_{56} = 0$ the equations governing the thickness-twist and the equations governing the face-shear are completely uncoupled, and hence the motions can occur at different frequencies, ω and $\bar{\omega}$.

LINEARLY TAPERED INFINITE STRIP

We consider the special case of an infinite strip with a linear taper in the x_3 direction; see Fig. 2. The taper is given by

$$h(x_3) = \frac{h_0}{l}x_3.$$

The edges $x_3 = 0$ and $x_3 = l$ are free, thus the boundary conditions, (4), become

$$\bar{u}' = \psi' = 0 \qquad \text{on} \qquad x_3 = 0, l. \tag{9}$$

Inserting the condition for the linear taper into (8) we obtain

$$\bar{u}'' + \frac{1}{x_3}\bar{u}' + \bar{\beta}^2\bar{u} = 0, \tag{10a}$$

$$\psi'' + \frac{3}{x_3}\psi' + \left(\beta^2 - \frac{\gamma^2 l^2}{h_0^2 x_3^2}\right)\psi = 0. \tag{10b}$$

The general solution of (10) is

$$\bar{u}(x_3) = AJ_0(\bar{\beta}x_3) + BY_0(\bar{\beta}x_3), \tag{11a}$$

$$\psi(x_3) = x_3^{-1}(CJ_p(\beta x_3) + DY_p(\beta x_3)), \tag{11b}$$

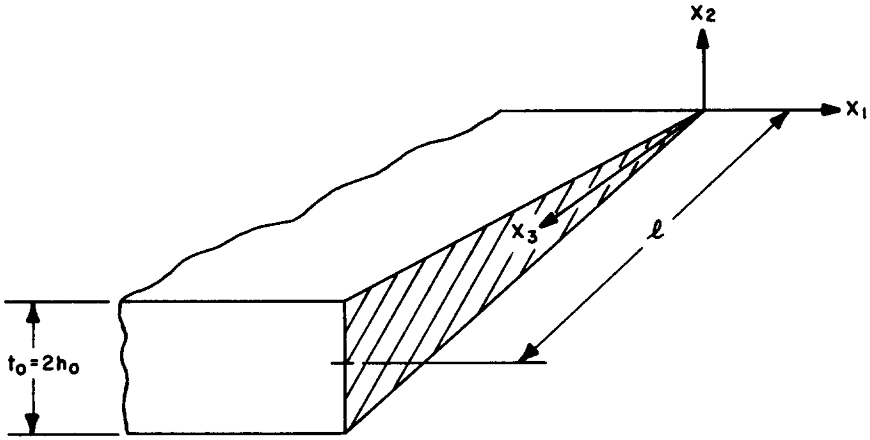

FIG. 2. Linearly tapered infinite strip.

where J_p and Y_p are Bessel functions of the first and the second kind respectively, A, B, C, D are constants, and

$$p^2 \equiv 1+\gamma^2 l^2/h_0^2. \tag{12}$$

Applying the boundary conditions, (9), we obtain $A = D = 0$ and the secular equations

$$J_1(\bar{\lambda}) = 0, \tag{13a}$$

$$\lambda J_{p-1}(\lambda)-(1+p)J_p(\lambda) = 0, \tag{13b}$$

where

$$\bar{\lambda} \equiv \bar{\beta} l, \qquad \lambda \equiv \beta l.$$

Introducing as a reference frequency the frequency, ω_s, of the lowest x_1, x_2 thickness-shear mode of a uniform plate of thickness $2h_0$, i.e.

$$\omega_s = \frac{\pi}{2h_0}\left(\frac{c_{66}}{\rho}\right)^{\frac{1}{2}},$$

we find for the nondimensional frequencies, $\bar{\Omega}_n$ and Ω_m, of the nth mode of face-shear and the mth mode of thickness-twist,

$$\bar{\Omega}_n \equiv \frac{\bar{\omega}_n}{\omega_s} = \frac{1}{\pi}\left(\frac{c_{55}}{c_{66}}\right)^{\frac{1}{2}}\left(\frac{t_0}{l}\right)\bar{\lambda}_n \qquad n = \text{integer}, \tag{14a}$$

$$\Omega_m \equiv \frac{\omega_m}{\omega_s} = \frac{1}{\pi}\left(\frac{c_{55}}{c_{66}}\right)^{\frac{1}{2}}\left(\frac{t_0}{l}\right)\lambda_m \qquad m = \text{integer}, \tag{14b}$$

where $t_0 \equiv 2h_0$ is the maximum thickness of the plate and $\bar{\lambda}_n$, λ_m are the successive roots of equations (13a) and (13b). In Fig. 3 the nondimensional frequencies, $\bar{\Omega}_n$ and Ω_m, are plotted as functions of the length-to-thickness ratio, l/t_0.

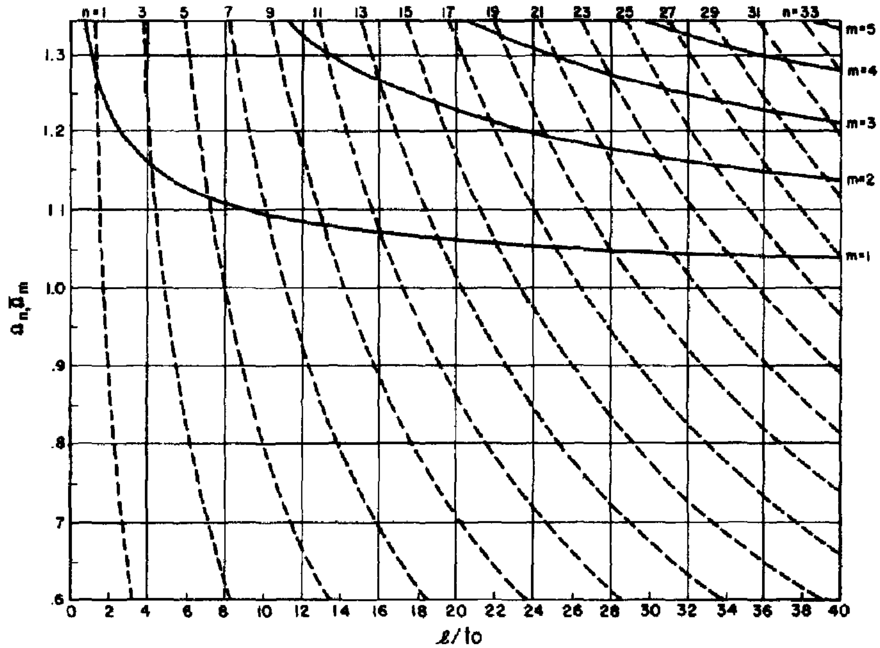

FIG. 3. Dimensionless frequencies of thickness-twist and face-shear modes in a linearly tapered plate as a function of length-to-thickness ratio.

COMPARISON WITH THE STRIP OF CONSTANT THICKNESS

In the case where $h(x_3) \equiv h_0$, a constant, equations (8) become

$$\bar{u}'' + \bar{\beta}^2 \bar{u} = 0, \tag{15a}$$

$$\psi'' + \left(\beta^2 - \frac{\gamma^2}{h_0^2}\right)\psi = 0. \tag{15b}$$

The solution of (15) can be written as

$$\bar{u} = A_c \sin \bar{\beta} x_3 + B_c \cos \bar{\beta} x_3,$$

$$\psi = C_c \sin \delta x_3 + D_c \cos \delta x_3,$$

where A_c, B_c, C_c, D_c are constants and

$$\delta^2 \equiv \beta^2 - \gamma^2/h_0^2.$$

Applying the boundary conditions (9) we obtain $A = C = 0$ and

$$\sin \bar{\beta} l = 0, \qquad \sin \delta l = 0,$$

from which

$$\bar{\beta}_q = q\pi/l \qquad q = \text{integer},$$

$$\delta_r = r\pi/l \qquad r = \text{integer}.$$

Again, introducing ω_s as a reference frequency we find for the nondimensional frequencies, $\bar{\Omega}_{qc}$ and Ω_{rc}, of the qth mode of face-shear and the rth mode of thickness-twist,

$$\bar{\Omega}_{qc} \equiv \frac{\bar{\omega}_{qc}}{\omega_s} = q\left(\frac{c_{55}}{c_{66}}\right)^{\frac{1}{2}}\left(\frac{t_0}{l}\right), \tag{16a}$$

$$\Omega_{rc} \equiv \frac{\omega_{rc}}{\omega_s} = \left[1+r^2\frac{c_{55}}{c_{66}}\left(\frac{t_0}{l}\right)^2\right]^{\frac{1}{2}} \tag{16b}$$

It is interesting to investigate the effects of the tapering by comparing the influence of the length-to-thickness ratio on the frequencies calculated from (16) and (14). For the face-shear modes, the comparison is easy to achieve. For large n, the nth root of (13a) is given approximately by

$$\bar{\lambda}_n = \pi(n+\tfrac{1}{4}),$$

and thus

$$\frac{\bar{\Omega}_n}{\bar{\Omega}_{qc}} = \frac{n+\frac{1}{4}}{q}. \tag{17}$$

We see that there is very little difference between the frequency of the kth mode ($k = n = q$) of face-shear in a tapered plate and its frequency in a uniform plate. This result is not surprising for the following reason. The solution for the face-shear vibrations of a uniform plate reveals that the frequencies of these vibrations are independent of the thickness of the plate. The same lack of explicit dependence on the thickness is exhibited by the frequencies, $\bar{\omega}_n$, of the face-shear modes of the tapered plate, since the solutions, $\bar{\lambda}_n$, of (13a) are independent of the thickness. Thus the only dependence on the thickness is a "shape effect" which introduces Bessel functions for a tapered plate instead of the circular functions which govern the motion in a uniform plate, and although the displacement pattern is altered by these functions, the frequencies remain substantially the same.

For the thickness-twist modes the effects of the tapering can not be readily isolated in analytical form because the length-to-thickness ratio is contained in the order, p, of the Bessel function, and, in addition, the roots of the secular equation, (13b), are not conveniently representable by a simple expression. Nevertheless, it is possible to calculate the effects numerically. In Table 2 the ratio, Ω_m/Ω_{rc}, of the nondimensional frequency of the linearly tapered plate to that of the constant thickness plate is given for m and $r = 1, 3, 6$ for several values of the length-to-thickness ratio.

TABLE 2. RATIO OF THE THICKNESS-TWIST FREQUENCIES OF A LINEARLY TAPERED PLATE TO THOSE OF A UNIFORM PLATE

l/t_0	Ω_m/Ω_{rc}		
	$m = r = 1$	$m = r = 3$	$m = r = 6$
20	1·0612	1·3183	1·5105
25	1·0539	1·2790	1·4641
30	1·0484	1·2492	1·4236
35	1·0441	1·2260	1·3893
40	1·0407	1·2073	1·3603

It is evident from Table 2 that the frequency of the lowest thickness-twist mode for the tapered plate is only slightly higher than that for the uniform plate whose thickness is equal to the maximum thickness of the tapered plate, and not twice as high as might be expected by assuming that the frequency spectrum of the tapered plate is comparable to that of a uniform plate whose thickness is equal to the average thickness of the tapered plate. It is also evident from Table 2 that the overtones of the thickness-twist modes of the tapered plate are spread out more than those of the uniform plate. Analogous behavior was noted by Mindlin and Forray [1] in their study of the thickness-shear and flexural vibrations of tapered plates, and an identical explanation can be offered to account for the present results. As will be shown in the following section, the fundamental thickness-twist motion is highly concentrated near the thick edge of the tapered plate and hence the motion is largely determined by the characteristics of the thick edge. For the overtones, however, a greater portion of the plate participates in the motion, and with this can be associated the spreading of the frequencies of the overtones.

MODE SHAPES

The nth mode of thickness-twist motion is described by

$$\psi_n = \psi_n^* l x_3^{-1} J_p(\lambda_n x_3/l)$$

where ψ_n^* is an arbitrary constant. In Fig. 4 several mode shapes, ψ_n/ψ_n^*, are plotted as a function of x_3/l, the nondimensional distance along the plate measured from the thin edge. It can be seen that for the fundamental mode the amplitude of the motion is negligible for nearly 80 per cent of the length of the plate. For the overtones, the maximum amplitude

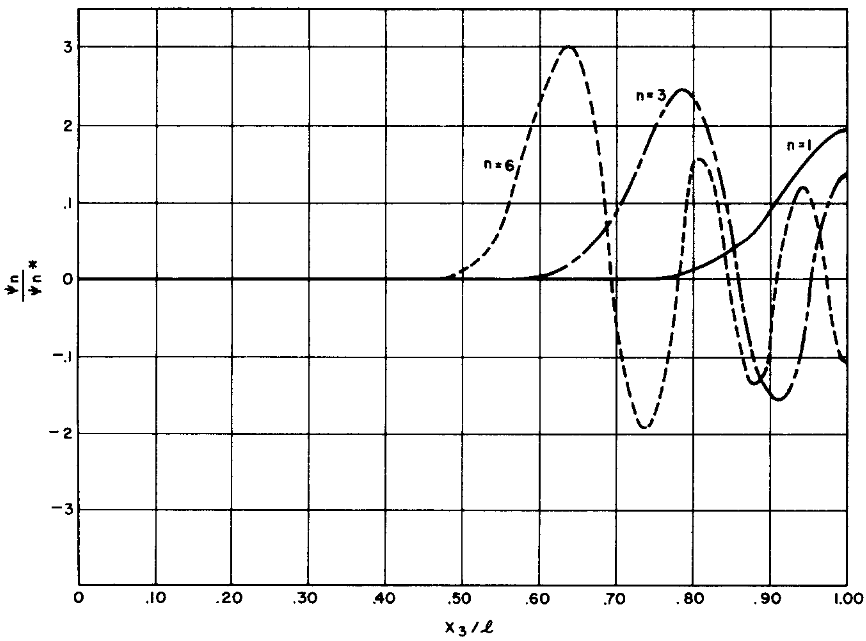

FIG. 4. Thickness-twist mode shapes for a linearly tapered plate with length-to-thickness ratio, $l/t_0 = 20$

of the motion moves in from the thick edge, and a greater portion of the plate participates in the motion.

The nth mode of face-shear motion is described by

$$\bar{u}_n = \bar{u}_n^* J_0(\bar{\lambda}_n x_3/l)$$

where $\bar{u}_n^*$ is an arbitrary constant. In Fig. 5 several mode shapes, $\bar{u}_n/\bar{u}_n^*$, are plotted as a function of x_3/l. These mode shapes are different in character from those of the thickness-twist motion. The motion in the fundamental mode and the low overtones of face-shear is more evenly distributed throughout the length of the plate although the maximum amplitude always occurs at the thin edge. In the higher overtones of face-shear, the motion becomes more and more localized near the thin edge of the plate.

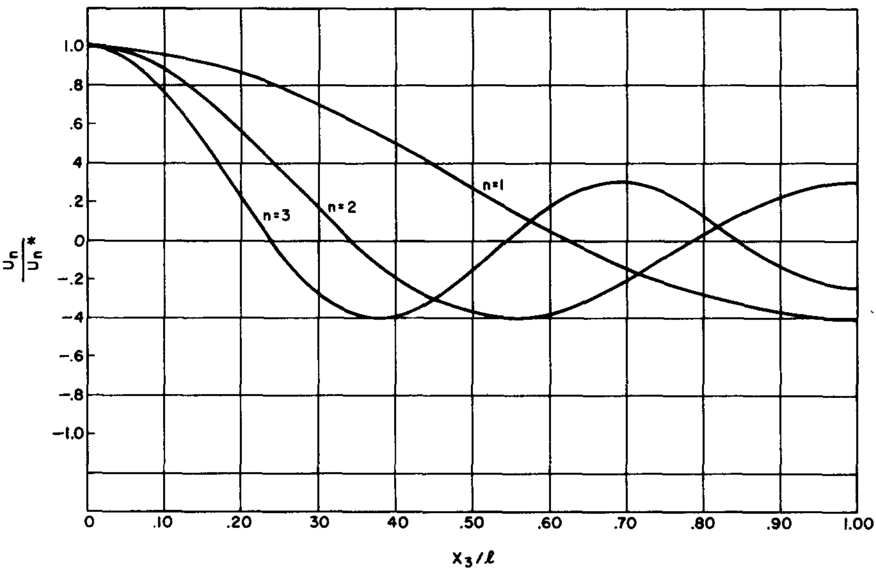

FIG. 5. Face-shear mode shapes for a linearly tapered plate.

Acknowledgement—The author would like to express his thanks to Mr. W. J. Denkmann of Bell Telephone Laboratories for generating the computer program used for the numerical computations.

REFERENCES

[1] R. D. MINDLIN and M. FORRAY, Thickness-shear and flexural vibrations of contoured crystal plates. *J. appl. Phys.* **25**, 12 (1954).
[2] R. P. JERRARD, Vibration of quartz crystal plates. *Q. appl. Math.* **18**, 173 (1960).
[3] R. D. MINDLIN and D. C. GAZIS, Strong resonances of rectangular AT-cut quartz plates. *Proc. 4th U.S. Natn. Congr. of Applied Mechanics*, pp. 305–310.
[4] R. D. MINDLIN, Thickness-twist vibrations of an infinite, monoclinic crystal plate. *Int. J. Solids Structures* **1**, 141 (1965).
[5] R. D. MINDLIN, High frequency vibrations of crystal plates. *Q. appl. Math.* **19**, 51 (1961).
[6] W. P. MASON, *Piezoelectric Crystals and Their Applications to Ultrasonics*. D. Van Nostrand, New York (1950).

(*Received* 1 *September* 1965)

Résumé—Des solutions sont obtenues pour les vibrations de torsion d'épaisseur et de cisaillement de surface dans une plaque de crystal d'une épaisseur linéaire variée. Les résultats sont comparés à ceux obtenus pour une plaque d'épaisseur constante. Il a été constaté que la fréquence du mode le plus bas de torsion d'épaisseur d'une plaque linéaire conique est seulement légèrement plus élevée que la fréquence du mode le plus bas de torsion d'épaisseur d'une plaque uniforme de la même épaisseur maximum. Il a été aussi constaté que les fréquences du mode de cisaillement de surface d'une plaque conique diffère légèrement de ceux d'une plaque uniforme.

Zusammenfassung—Lösungen für die Dickenverdrehung und für Flächenscherungs Schwingungen in einer Kristallplatte von linear veränderlicher Dicke werden erhalten. Die Ergebnisse werden mit denen von einer Platte von beständiger Dicke verglichen. Es wurde gefunden, das die Frequenz der niedrigsten Dickenverdrehungs Anordnung einer konischen linearen Platte ist nur etwas höher als die Frequenz der niedrigsten Dickenverdrehungs Form einer gleichförmigen Platte derselben maximalen Dicke. Es wurde also gefunden, das die Frequenzen der Flächenscherungs Formen in einer konischen Platte nur sehr wenig verschieden sind von denen einer gleichförmigen Platte.

Абстракт—Получены решения для колебаний в зависимости от толщины-кручения и поверхности-сдвига в кристаллической пластине с линейно изменяющейся толщиной. Результаты сравниваются с теми результатами, которые получены для пластины с постоянной толщиной. Найдено, что частота формы с самой малой толщиной кручения линейно конической пластины только слегка превышает частоту формы самой малой толщины-кручения однородной пластины такой же максимальной толщины. Найдено, также, что частоты форм поверхностного-сдвига в конической пластине очень мало отличаются от частот в однородной пластине.